Math 150, *Lecture Notes- Bonds* Name____________

Section 1.3 *Evaluating Limits Analytically*

Properties of Limits:

THEOREM 1.1 Some Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x\to c} b = b$ **2.** $\lim_{x\to c} x = c$ **3.** $\lim_{x\to c} x^n = c^n$

THEOREM 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. Scalar multiple: $\lim_{x\to c} [b\,f(x)] = bL$
2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

Use ***basic limits and properties of limits*** to find the following limits.

Ex.1 $\lim_{x\to 1}\left(12x^3 - 6x + 5\right)$

Ex.2 $\lim_{x \to 1} \frac{3x+5}{x+1}$

THEOREM 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

THEOREM 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

THEOREM 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

Ex.3 $\lim_{x \to -3} \sqrt[3]{12x+3}$

Ex.4 Given $\lim_{x \to c} f(x) = 27$, evaluate the following limits:

(a) $\lim_{x \to c} \sqrt[3]{f(x)} =$

(b) $\lim_{x \to c} \frac{f(x)}{18} =$

(c) $\lim_{x \to c} [f(x)]^2 =$

(d) $\lim_{x \to c} [f(x)]^{\frac{2}{3}} =$

THEOREM 1.6 Limits of Trigonometric Functions

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to c} \sin x = \sin c$
2. $\lim_{x \to c} \cos x = \cos c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \cot x = \cot c$
5. $\lim_{x \to c} \sec x = \sec c$
6. $\lim_{x \to c} \csc x = \csc c$

Ex.5 Evaluate: $\lim_{x \to \frac{5\pi}{3}} \cos(x)$

THEOREM 1.7 Functions That Agree at All But One Point

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

Ex.6 Evaluate: $\lim_{x \to -2} \frac{3x^2 + 5x - 2}{x + 2}$

A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. If the limit of $f(x)$ as x approaches c *cannot* be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. [Choose g such that the limit of $g(x)$ *can* be evaluated by direct substitution.]
3. Apply Theorem 1.7 to conclude *analytically* that

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c).$$

4. Use a *graph* or *table* to reinforce your conclusion.

Ex.7 Evaluate: $\lim\limits_{x \to 3} \dfrac{\sqrt{x+1} - 2}{x - 3}$

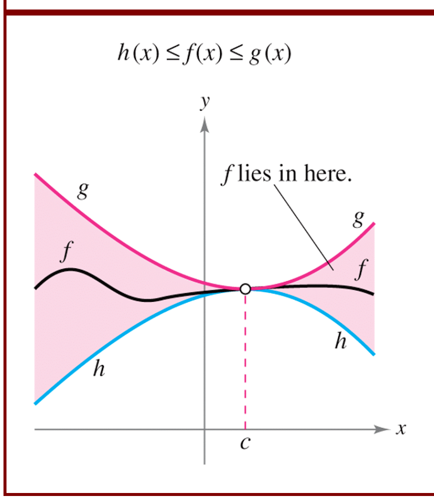
Ex.8 Evaluate: $\lim_{x \to 0} \dfrac{\dfrac{1}{x+4} - \dfrac{1}{4}}{x}$

THEOREM 1.8 The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is equal to L.



Ex.9 Evaluate: $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right)$

THEOREM 1.9 Two Special Trigonometric Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ **2.** $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Ex.10 Evaluate: $\lim_{x \to 0} \frac{\cos(x) - \sin(x) - 1}{2x}$

Ex.11 Given $f(x) = 3x^2 + x$, evaluate: $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$